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q-deformed Lorentz-algebra in Minkowski phase space

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Abstract. In the present paper we show that the Lorentz algebra $\mathcal{L}$ as defined in [5] is isomorphic to an algebra $\hat{\mathcal{U}}$ closely related to a q-deformed $SU_q(2) \otimes SU_q(2)$ algebra. On this algebra $\hat{\mathcal{U}}$ we define a Hopf algebra structure and show its action on q-spinor modules. This algebra is related to the q-deformed Minkowski space algebra by a non invertible factorisation.

Introduction

The q-deformed Lorentz [4,5] and Poincare algebra [6] has been studied in previous papers. The concept of non-commutative coordinates in a four-dimensional Minkowski space has been introduced in [9,10], based on a q-Lorentz group-covariant q-deformation of the Heisenberg algebra. For the very definition of this algebra orbital q-deformed angular momentum as well as a scaling operation had to be introduced.

As usual, orbital angular momentum restricts the representations as it has no spinor representations. In the q-deformed version of the algebra it is convenient to include the Casimir operator in the defining relations of the algebra. A restriction of the representations can be expressed through conditions on the Casimir operators and thus leads to a non-invertible factorization of the algebra. This factorization has been studied in [10]. For this purpose a very convenient definition of the q-Lorentz algebra has been found that exhibits the close relation of the algebra to a deformed $SU_q(2) \otimes SU_q(2)$ algebra.

It seems natural to start from this definition of the q-deformed Lorentz algebra and we shall do so in this paper. We shall show that there exists a Hopf algebra isomorphism between this algebra and the Lorentz algebra defined in [4,5]. Then we study the four-vector-like modules of the new algebra, relate them to the q-Poincare algebra and finally to the q-deformed Minkowsky space algebra [10]. It clearly shows the module structure of the Minkowski space as a q-Lorentz algebra module, whereas the previous treatment was based on a q-Lorentz group comodule structure. This can now be generalized to spinorial modules as well.

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1 q-Lorentz algebra

The q-Lorentz algebra can be defined in close analogy to the $SU_q(2) \times SU_q(2)$ splitting of $SO(4)$ if we use eight generators [3], this follows Pauli's treatment of $SO(3,1)$ [3].

The $SU_q(2)$ algebra can be defined with four generators [9,10]:

$$\varepsilon_{BC}{}^{A} L^C L^B = -\frac{1}{q^2} W L^A \tag{1.1}$$
$$q^6 \lambda^2 (L \circ L) = W^2 - 1$$

The ε-symbol is the q-deformed antisymmetric tensor and the scalar product $\circ$ is defined with the q-deformed metric g_{AB}. Both are given in the Appendix. $\lambda = q - \frac{1}{q}$.

This version of $SU_q(2)$ is related to the standard version by the transformation:

$$\begin{aligned} T^+ &:= q^{\frac{5}{2}} [2]^{\frac{1}{2}} \tau^{\frac{1}{2}} L^+ \\ T^- &:= -q^{\frac{7}{2}} [2]^{\frac{1}{2}} \tau^{\frac{1}{2}} L^- \\ (\tau^3)^{-\frac{1}{2}} &:= W - q^3 \lambda L^3, \end{aligned} \tag{1.2}$$

where $\tau^{\frac{1}{2}}$ is the inverse of $W - q^3 \lambda L^3$.

For τ^3, T^+, T^- the standard q-commutation relation follows from (1.1).

$$\begin{aligned} \tau^3 T^+ &= q^{-4} T^+ \tau^3 \\ \tau^3 T^- &= q^4 T^- \tau^3 \\ T^+ T^- &= q^2 T^- T^+ + q \lambda^{-1} (1 - \tau^3) \end{aligned} \tag{1.3}$$

This version of $SU_q(2)$ can be generalized to the q-Lorentz algebra:

$$\varepsilon_{CB}{}^{A} \hat{R}^B \hat{R}^C = \frac{1}{q[2]} \hat{U} \hat{R}^A \tag{1.4}$$

$$\varepsilon_{CB}{}^{A}\hat{S}^{B}\hat{S}^{C} = -\frac{1}{q[2]}\hat{U}'\hat{S}^{A}$$

$$\hat{R}^{A}\hat{S}^{B} = q^{2}\hat{R}^{AB}{}_{CD}\hat{S}^{C}\hat{R}^{D}$$

$$q^{4}[2]^{2}\lambda^{2}\hat{R}\circ\hat{R} = \hat{U}^{2}-1$$
$$q^{4}[2]^{2}\lambda^{2}\hat{S}\circ\hat{S} = \hat{U}'^{2}-1$$

The $\hat{R}$-matrix as well as the definition of the $[n]$ symbol can be found in the Appendix.

The elements $\hat{U}$, $\hat{U}'$ are central, which is consistent with the last two equations in (1.4). They are the two Casimir operators of the q-Lorentz algebra and the last two equations in (1.4) reduce the number of independent generators to the six generators $\hat{R^{A}}$ and $\hat{S^{A}}$.

It is only the sign in the $\hat{R}\hat{R}$ relations that differs from the $SU_q(2)$ relations (1.1). This is due to the non-compact nature of $SO(3,1)$. For the $\hat{R}\hat{S}$ relations covariance and the Poincare-Birkhoff-Witt property demands the above structure. The algebra (1.4) is compatible with the conjugation properties

$$\begin{aligned}\overline{\hat{U}} &= \hat{U}' \\ \overline{\hat{R}^{A}} &= -g_{AB}\hat{S}^{B} \\ \overline{\hat{S}^{A}} &= -g_{AB}\hat{R}^{B}\end{aligned} \tag{1.5}$$

The algebraic relations (1.4) and (1.5) define a q-Lorentz algebra. To verify this statement we first show that the seven-generator version of the q-Lorentz algebra [5], which we shall call the $\mathcal{L}$-algebra, can be mapped into our algebra which we shall call the $\hat{\mathcal{U}}$-algebra. The algebra morphism Ψ: $\mathcal{L}\to\hat{\mathcal{U}}$ is analogous to the morphism given in (1.2). We first identify the "diagonal" $SU_q(2)$ part that is isomorphic to the algebra (1.1):

$$\begin{aligned}\hat{L}^{A} &= \frac{[2]^{2}}{q}\left(\hat{U}\hat{S}^{A}-\hat{U}'\hat{R}^{A}+q^{2}\lambda[2]\varepsilon_{CB}{}^{A}\hat{R}^{B}\hat{S}^{C}\right) \\ \hat{W} &= \hat{U}\hat{U}'-q^{6}\lambda^{2}[2]^{2}\left(\hat{R}\circ\hat{S}\right)\end{aligned} \tag{1.6}$$

This is a generalization of the algebra automorphism found in [10] and was first found in [12].

From (1.2) follows the identification of the $SU_q(2)$ part of Ψ: $\mathcal{L}\to\hat{\mathcal{U}}$:

$$\begin{aligned}T^{+} &:= q^{\frac{5}{2}}[2]^{\frac{1}{2}}\hat{\tau}^{\frac{1}{2}}\hat{L}^{+} \\ T^{-} &:= -q^{\frac{7}{2}}[2]^{\frac{1}{2}}\hat{\tau}^{\frac{1}{2}}\hat{L}^{-} \\ \tau^{3} &:= (\hat{\tau}^{\frac{1}{2}})^{2}\end{aligned} \tag{1.7}$$

where $\hat{\tau}^{\frac{1}{2}}$ is the inverse of $\hat{W}-q^{3}\lambda\hat{L}^{3}$. For the remaining four generators of $\mathcal{L}$ we found:

$$\begin{aligned}T^{2} &:= q^{\frac{1}{2}}[2]^{\frac{3}{2}}\hat{R}^{+} \\ \tau^{1} &:= -q^{2}\lambda[2]\hat{R}^{3}-\hat{U} \\ S^{1} &:= -q^{\frac{3}{2}}[2]^{\frac{3}{2}}\hat{\tau}^{\frac{1}{2}}\hat{S}^{-} \\ \sigma^{2} &:= \hat{\tau}^{\frac{1}{2}}(q^{2}\lambda[2]\hat{S}^{3}-\hat{U}')\end{aligned} \tag{1.8}$$

The algebraic relations of $\mathcal{L}$ as they were given in [5] follow from the relations (1.4). This establishes the algebra morphism of Ψ: $\mathcal{L}\to\hat{\mathcal{U}}$. It also preserves the conjugation properties.

The inverse of this morphism can also be found. For Φ: $\hat{\mathcal{U}}\to\mathcal{L}$ we have:

$$\begin{aligned}\hat{R}^{+} &:= q^{-\frac{1}{2}}[2]^{-\frac{3}{2}}T^{2} \\ \hat{R}^{-} &:= -q^{-\frac{5}{2}}[2]^{-\frac{3}{2}}\left(qS^{1}+\tau^{1}T^{-}\right) \\ \hat{R}^{3} &:= \frac{\lambda}{q^{2}[2]^{2}}T^{2}T^{-}+\frac{1}{q\lambda[2]^{2}}\left(\sigma^{2}-\tau^{1}\right) \\ \hat{U} &:= -\frac{\lambda^{2}}{[2]}T^{2}T^{-}-\frac{q}{[2]}\sigma^{2}-\frac{1}{q[2]}\tau^{1}\end{aligned} \tag{1.9}$$

$$\begin{aligned}\hat{S}^{-} &:= -q^{-\frac{3}{2}}[2]^{-\frac{3}{2}}\left(\tau^{3}\right)^{-\frac{1}{2}}S^{1} \\ \hat{S}^{+} &:= q^{-\frac{3}{2}}[2]^{-\frac{3}{2}}\left(\tau^{3}\right)^{-\frac{1}{2}}\left(q\tau^{3}T^{2}-\sigma^{2}T^{+}\right) \\ \hat{S}^{3} &:= \left(\tau^{3}\right)^{-\frac{1}{2}}\left(\frac{\lambda}{[2]^{2}}S^{1}T^{+}+\frac{1}{q^{3}\lambda[2]^{2}}\left(\sigma^{2}-\tau^{3}\tau^{1}\right)\right) \\ \hat{U}' &:= \left(\tau^{3}\right)^{-\frac{1}{2}}\left(\frac{q^{2}\lambda^{2}}{[2]}S^{1}T^{+}-\frac{q}{[2]}\sigma^{2}-\frac{1}{q[2]}\tau^{3}\tau^{1}\right)\end{aligned} \tag{1.10}$$

The detailed verification that this is the desired *algebra homomorphism Φ is tedious.

2 Hopf algebra structure

For $\mathcal{L}$ a Hopf algebra structure was defined in [5]. It is possible to carry this structure on $\hat{\mathcal{U}}$. To write the comultiplication in a more compact form we define the elements $\hat{\rho}$ and $\hat{\sigma}$ of $\hat{\mathcal{U}}$:

$$\begin{aligned}\hat{\rho} &= q^{2}\lambda[2]\hat{R}^{3}+\hat{U} \\ \hat{\sigma} &= q^{2}\lambda[2]\hat{S}^{3}-\hat{U}'\end{aligned} \tag{2.11}$$

First the counit ε; we also list $\varepsilon(\hat{\rho})$ and $\varepsilon(\hat{\sigma})$:

$$\begin{aligned}\varepsilon(\hat{R}^{+}) &= 0 \\ \varepsilon(\hat{R}^{-}) &= 0 \\ \varepsilon(\hat{R}^{3}) &= 0 \\ \varepsilon(\hat{U}) &= -1 \\ \varepsilon(\hat{\rho}) &= -1\end{aligned} \tag{2.12}$$

$$\begin{aligned}\varepsilon(\hat{S}^{+}) &= 0 \\ \varepsilon(\hat{S}^{-}) &= 0 \\ \varepsilon(\hat{S}^{3}) &= 0 \\ \varepsilon(\hat{U}') &= -1 \\ \varepsilon(\hat{\sigma}) &= 1\end{aligned} \tag{2.13}$$

The coproduct Δ:

$$\Delta\left(\hat{R}^{+}\right) = \hat{\sigma}\otimes\hat{R}^{+}-\hat{R}^{+}\otimes\hat{\rho} \tag{2.14}$$

$$
\begin{aligned}
\Delta\left(\hat{R}^-\right) &= \left(\hat{\tau}^{\frac{1}{2}} \otimes \hat{\tau}^{\frac{1}{2}}\right)\left[\frac{q}{[2]}\hat{\rho}\hat{L}^- \otimes \hat{\tau}^{-\frac{1}{2}}\hat{\rho}\right. \\
&\quad + \hat{S}^- \otimes \left(\hat{\sigma} - q^5\lambda^2[2]^2\hat{R}^+\hat{L}^-\right) \\
&\quad + \hat{\rho} \otimes \left(\frac{q}{[2]}\hat{\rho}\hat{L}^- - \hat{S}^-\right) \\
&\quad \left. - q^3\lambda^2[2]^2\hat{S}^-\hat{L}^- \otimes \hat{\tau}^{-\frac{1}{2}}\hat{R}^+\right] \\
\Delta\left(\hat{R}^3\right) &= -\hat{R}^3 \otimes \hat{\rho} + \hat{\tau}^{\frac{1}{2}}\hat{\sigma} \otimes \hat{R}^3 \\
&\quad + q^3\lambda[2]\hat{S}^- \otimes \hat{R}^+ \\
&\quad - q^3\lambda[2]\hat{\tau}^{\frac{1}{2}}\hat{R}^+ \otimes \hat{R}^- \\
&\quad - q^2\lambda\hat{\tau}^{\frac{1}{2}}\hat{\sigma}\hat{L}^- \otimes \hat{R}^+ \\
\Delta\left(\hat{U}\right) &= -\hat{U} \otimes \hat{\rho} + q^3\lambda^2[2]^2\hat{S}^- \otimes \hat{R}^+ \\
&\quad - q^2\lambda[2]\hat{\tau}^{\frac{1}{2}}\hat{\sigma} \otimes \hat{R}^3 \\
&\quad + q^5\lambda^2[2]^2\hat{\tau}^{\frac{1}{2}}\hat{R}^+ \otimes \hat{R}^- \\
&\quad + q^4\lambda^2[2]\hat{\tau}^{\frac{1}{2}}\hat{\sigma}\hat{L}^- \otimes \hat{R}^+ \\
\Delta(\hat{\rho}) &= -\hat{\rho} \otimes \hat{\rho} + q^4\lambda^2[2]^3\hat{S}^- \otimes \hat{R}^+
\end{aligned}
$$

$$
\begin{aligned}
\Delta\left(\hat{S}^-\right) &= \hat{S}^- \otimes \hat{\sigma} - \hat{\rho} \otimes \hat{S}^- \qquad (2.15) \\
\Delta(\hat{S}^+) &= (\hat{\tau}^{\frac{1}{2}} \otimes \hat{\tau}^{\frac{1}{2}})\left[-\frac{q}{[2]}\hat{\sigma}\hat{L}^+ \otimes \hat{\tau}^{-\frac{1}{2}}\hat{\sigma}\right. \\
&\quad + \hat{R}^+ \otimes \left(q^5\lambda^2[2]^2\hat{S}^-\hat{L}^+ - \hat{\rho}\right) \\
&\quad + \hat{\sigma} \otimes \left(\hat{R}^+ - \frac{q}{[2]}\hat{\sigma}\hat{L}^+\right) \\
&\quad \left. + q^7\lambda^2[2]^2\hat{R}^+\hat{L}^+ \otimes \hat{\tau}^{-\frac{1}{2}}\hat{S}^-\right] \\
\Delta\left(\hat{S}^3\right) &= \hat{S}^3 \otimes \hat{\sigma} - \hat{\tau}^{\frac{1}{2}}\hat{\rho} \otimes \hat{S}^3 \\
&\quad - q\lambda[2]\hat{R}^+ \otimes \hat{S}^- \\
&\quad + q\lambda[2]\hat{\tau}^{\frac{1}{2}}\hat{S}^- \otimes \hat{S}^+ \\
&\quad + q^4\lambda\hat{\tau}^{\frac{1}{2}}\hat{\rho}\hat{L}^+ \otimes \hat{S}^- \\
\Delta\left(\hat{U}'\right) &= \hat{U}' \otimes \hat{\sigma} + q^5\lambda^2[2]^2\hat{R}^+ \otimes \hat{S}^- \\
&\quad - q^2\lambda[2]\hat{\tau}^{\frac{1}{2}}\hat{\rho} \otimes \hat{S}^3 + \\
&\quad + q^3\lambda^2[2]^2\hat{\tau}^{\frac{1}{2}}\hat{S}^- \otimes \hat{S}^+ + \\
&\quad + q^6\lambda^2[2]\hat{\tau}^{\frac{1}{2}}\hat{\rho}\hat{L}^+ \otimes \hat{S}^- \\
\Delta(\hat{\sigma}) &= \hat{\sigma} \otimes \hat{\sigma} - q^4\lambda^2[2]^3\hat{R}^+ \otimes \hat{S}^-
\end{aligned}
$$

Finally the antipode:

$$
\begin{aligned}
S\left(\hat{R}^+\right) &= -q^2\hat{\tau}^{\frac{1}{2}}\hat{R}^+ \qquad (2.16) \\
S\left(\hat{R}^-\right) &= -\hat{S}^- - \frac{1}{q[2]}\hat{\tau}^{\frac{1}{2}}\hat{L}^-\hat{\sigma} \\
S\left(\hat{R}^3\right) &= -\frac{1}{q^2\lambda[2]}\hat{U} - \frac{1}{q^2\lambda[2]}\hat{\tau}^{\frac{1}{2}}\hat{\sigma} \\
S\left(\hat{U}\right) &= \hat{U} \\
S(\hat{\rho}) &= -\hat{\tau}^{\frac{1}{2}}\hat{\sigma} \\
S\left(\hat{S}^+\right) &= -\hat{R}^+ - \frac{q^3}{[2]}\hat{\tau}^{\frac{1}{2}}\hat{L}^+\hat{\rho} \\
S\left(\hat{S}^-\right) &= -\frac{1}{q^2}\hat{\tau}^{\frac{1}{2}}\hat{S}^- \\
S\left(\hat{S}^3\right) &= \frac{1}{q^4\lambda[2]}\hat{U}' - \frac{1}{q^2\lambda[2]}\hat{\tau}^{\frac{1}{2}}\hat{\rho} \\
S\left(\hat{U}'\right) &= \hat{U}' \\
S(\hat{\sigma}) &= -\hat{\tau}^{\frac{1}{2}}\rho
\end{aligned}
$$

This establishes $\hat{\mathcal{U}}$ as a Hopf algebra.

3 $\hat{\mathcal{U}}$ module structures

In [5] a spinor module over the Hopfalgebra $\mathcal{L}$ was introduced. Since the Hopfalgebra $\hat{\mathcal{U}}$ is isomorphic to $\mathcal{L}$ as we have seen in the previous section an equivalent action of $\hat{\mathcal{U}}$ on the spinor module can be calculated. The results are:

$$
\begin{aligned}
\hat{R}^+x &= x\hat{R}^+ - q^{-\frac{1}{2}}[2]^{-\frac{3}{2}}y\hat{\rho} \qquad (3.17) \\
\hat{R}^+\overline{x} &= q\overline{x}\hat{R}^+ \\
\hat{R}^+y &= y\hat{R}^+ \\
\hat{R}^+\overline{y} &= \frac{1}{q}\overline{y}\hat{R}^+
\end{aligned}
$$

$$
\begin{aligned}
\hat{R}^-x &= x\hat{R}^- \qquad (3.18) \\
\hat{R}^-\overline{x} &= \frac{1}{q}\overline{x}\hat{R}^- - \lambda q^{-\frac{1}{2}}[2]^{\frac{1}{2}}\overline{y}\hat{R}^3 \\
\hat{R}^-y &= y\hat{R}^- + q^{-\frac{7}{2}}[2]^{-\frac{3}{2}}x\hat{\rho} \\
\hat{R}^-\overline{y} &= q\overline{y}\hat{R}^-
\end{aligned}
$$

$$
\begin{aligned}
\hat{R}^3x &= \frac{2}{[2]}x\hat{R}^3 - \frac{1}{q[2]^2}x\hat{U} - \lambda q^{\frac{3}{2}}[2]^{-\frac{1}{2}}y\hat{R}^- \qquad (3.19) \\
\hat{R}^3\overline{x} &= \overline{x}\hat{R}^3 - \lambda q^{-\frac{1}{2}}[2]^{\frac{1}{2}}\overline{y}\hat{R}^+ \\
\hat{R}^3y &= \frac{2}{[2]}y\hat{R}^3 + \frac{1}{q^3[2]^2}y\hat{U} + \lambda q^{-\frac{3}{2}}[2]^{-\frac{1}{2}}x\hat{R}^+ \\
\hat{R}^3\overline{y} &= \overline{y}\hat{R}^3
\end{aligned}
$$

$$
\begin{aligned}
\hat{U}x &= \frac{[4]}{[2]^2}x\hat{U} - q\lambda^2x\hat{R}^3 + \lambda^2q^{\frac{7}{2}}[2]^{\frac{1}{2}}y\hat{R}^- \qquad (3.20) \\
\hat{U}\overline{x} &= \overline{x}\hat{U} \\
\hat{U}y &= \frac{[4]}{[2]^2}y\hat{U} + q^3\lambda^2y\hat{R}^3 - \lambda^2q^{\frac{1}{2}}[2]^{\frac{1}{2}}x\hat{R}^+ \\
\hat{U}\overline{y} &= \overline{y}\hat{U}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}x &= \frac{1}{q}x\hat{\rho} \qquad (3.21) \\
\hat{\rho}\overline{x} &= \overline{x}\hat{\rho} - \lambda^2q^{\frac{3}{2}}[2]^{\frac{3}{2}}\overline{y}\hat{R}^+ \\
\hat{\rho}y &= qy\hat{\rho}
\end{aligned}
$$

$$\hat{\rho}\overline{y} = \overline{y}\hat{\rho}$$

$$\begin{aligned}\hat{S}^+ x &= qx\hat{S}^+ - \lambda q^{\frac{3}{2}}[2]^{\frac{3}{2}} y\hat{S}^3 \\ \hat{S}^+\overline{x} &= \overline{x}\hat{S}^+ \\ \hat{S}^+ y &= \frac{1}{q} y\hat{S}^+ \\ \hat{S}^+\overline{y} &= \overline{y}\hat{S}^+ + q^{-\frac{3}{2}}[2]^{-\frac{3}{2}}\overline{x}\hat{\sigma}\end{aligned} \tag{3.22}$$

$$\begin{aligned}\hat{S}^- x &= \frac{1}{q} x\hat{S}^- \\ \hat{S}^-\overline{x} &= \overline{x}\hat{S}^- - q^{-\frac{5}{2}}[2]^{-\frac{3}{2}}\overline{y}\hat{\sigma} \\ \hat{S}^- y &= qy\hat{S}^- \\ \hat{S}^-\overline{y} &= \overline{y}\hat{S}^-\end{aligned} \tag{3.23}$$

$$\begin{aligned}\hat{S}^3 x &= x\hat{S}^3 - \lambda q^{\frac{3}{2}}[2]^{\frac{1}{2}} y\hat{S}^- \\ \hat{S}^3\overline{x} &= \frac{2}{[2]}\overline{x}\hat{S}^3 - \frac{1}{q^3[2]^2}\overline{x}\hat{U}' - \lambda q^{-\frac{1}{2}}[2]^{-\frac{1}{2}}\overline{y}\hat{S}^- \\ \hat{S}^3 y &= y\hat{S}^3 \\ \hat{S}^3\overline{y} &= \frac{2}{[2]}\overline{y}\hat{S}^3 + \frac{1}{q[2]^2}\overline{y}\hat{U}' + \lambda q^{\frac{1}{2}}[2]^{-\frac{1}{2}}\overline{x}\hat{S}^-\end{aligned} \tag{3.24}$$

$$\begin{aligned}\hat{U}' x &= x\hat{U}' \\ \hat{U}'\overline{x} &= \frac{[4]}{[2]^2}\overline{x}\hat{U}' - q^3\lambda^2\overline{x}\hat{S}^3 - \lambda^2 q^{\frac{3}{2}}[2]^{\frac{1}{2}}\overline{y}\hat{S}^- \\ \hat{U}' y &= y\hat{U}' \\ \hat{U}'\overline{y} &= \frac{[4]}{[2]^2}\overline{y}\hat{U}' + q\lambda^2\overline{y}\hat{S}^3 + \lambda^2 q^{\frac{5}{2}}[2]^{\frac{1}{2}}\overline{y}\hat{S}^-\end{aligned} \tag{3.25}$$

$$\begin{aligned}\hat{\sigma} x &= x\hat{\sigma} - \lambda^2 q^{\frac{7}{2}}[2]^{\frac{1}{2}} y\hat{S}^- \\ \hat{\sigma}\overline{x} &= q\overline{x}\hat{\sigma} \\ \hat{\sigma} y &= y\hat{\sigma} \\ \hat{\sigma}\overline{y} &= \frac{1}{q}\overline{y}\hat{\sigma}\end{aligned} \tag{3.26}$$

We know how the algebra $\mathcal{L}$ acts on module spaces. Starting from spinor modules all the finite dimensional modules can be constructed. We are interested in the Minkowski module representing four-dimensional space time or the energy momentum variables P^a as well. The $\mathcal{L}$ module structure implies a $\hat{\mathcal{U}}$ module.

The algebraic structure of the four-vector space compatible with the comodule structure is:

$$\begin{aligned}P^0 P^A &= P^A P^0 \\ \varepsilon_{CB}{}^A P^B P^C &= -q\lambda P^0 P^A\end{aligned} \tag{3.27}$$

On this space, $\hat{\mathcal{U}}$ acts as follows:

$$\begin{aligned}\hat{R}^A P^0 &= \frac{[4]}{[2]^2} P^0\hat{R}^A \\ &\quad - \frac{1}{q[2]^2} P^A\hat{U} + \frac{\lambda}{q[2]}\varepsilon_{CB}{}^A P^B\hat{R}^C \\ \hat{R}^A P^B &= \frac{1}{q[2]}\Big[q^2[2]P^A\hat{R}^B - \lambda\varepsilon_C{}^{AB}P^0\hat{R}^C \\ &\quad - \lambda g^{AB}(P\circ\hat{R}) - \frac{1}{q^2[2]}g^{AB}P^0\hat{U} \\ &\quad - \frac{2}{q}\varepsilon_C{}^{AB}\varepsilon_{ST}{}^C P^T\hat{R}^S + \frac{1}{q^2[2]}\varepsilon_C{}^{AB}P^C\hat{U}\Big]\end{aligned} \tag{3.28}$$

$$\begin{aligned}\hat{S}^A P^0 &= \frac{[4]}{[2]^2} P^0\hat{S}^A \\ &\quad - \frac{1}{q^3[2]^2} P^A\hat{U}' + \frac{\lambda}{q[2]}\varepsilon_{CB}{}^A P^B\hat{S}^C \\ \hat{S}^A P^B &= \frac{1}{q[2]}\Big[[2]P^A\hat{S}^B - \lambda\varepsilon_C{}^{AB}P^0\hat{S}^C \\ &\quad + q^2\lambda g^{AB}(P\circ\hat{S}) - \frac{1}{[2]}g^{AB}P^0\hat{U}' \\ &\quad - \frac{2}{q}\varepsilon_C{}^{AB}\varepsilon_{ST}{}^C P^T\hat{S}^S - \frac{1}{q^2[2]}\varepsilon_C{}^{AB}P^C\hat{U}'\Big]\end{aligned} \tag{3.29}$$

$$\begin{aligned}\hat{U}P^0 &= \frac{[4]}{[2]^2}P^0\hat{U} - q\lambda^2(P\circ\hat{R}) \\ \hat{U}P^A &= \frac{[4]}{[2]^2}P^A\hat{U} - q^3\lambda^2 P^0\hat{R}^A - q\lambda^2\varepsilon_{CB}{}^A P^B\hat{R}^C\end{aligned} \tag{3.30}$$

$$\begin{aligned}\hat{U}'P^0 &= \frac{[4]}{[2]^2}P^0\hat{U}' - q^3\lambda^2(P\circ\hat{S}) \\ \hat{U}'P^A &= \frac{[4]}{[2]^2}P^A\hat{U}' - q\lambda^2 P^0\hat{S}^A + q\lambda^2\varepsilon_{CB}{}^A P^B\hat{S}^C\end{aligned} \tag{3.31}$$

These relations are consistent with the conjugation property:

$$\overline{P^0} = P^0, \qquad \overline{P^A} = g_{AB}P^B \tag{3.32}$$

The invariant "length" of a four-vector is:

$$P^2 = -P^0P^0 + P\circ P =: \eta_{ab}P^aP^b \tag{3.33}$$

This defines the four-metric η_{ab}. Invariance means:

$$AP^2 = P^2A, \quad \text{for} \quad A\in\hat{\mathcal{U}} \tag{3.34}$$

This again justifies to call the $\hat{\mathcal{U}}$ algebra q-Lorentz algebra.

It is useful to know the action of the $\hat{L}$ algebra defined in (1.6) on P^a:

$$\begin{aligned}\hat{L}^A P^B &= g^{AB}(P\circ\hat{L}) \\ &\quad - \frac{1}{q^4}\varepsilon_C{}^{AB}P^C\hat{W} - \frac{1}{q^2}\varepsilon_{CM}{}^A\varepsilon_N{}^{CB}P^M\hat{L}^N \\ \hat{W}P^A &= \left(\frac{q^4-q^2+1}{q^2}\right)P^A\hat{W} + \left(q^2-1\right)^2\varepsilon_{BC}{}^A P^C\hat{L}^B\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\hat{L}^A P^0 &= P^0\hat{L}^A \\ \hat{W}P^0 &= P^0\hat{W}\end{aligned} \tag{3.36}$$

Equation (3.36) shows that the 0-component of a four-vector is left invariant by $\hat{L}$. This again justifies to call them rotations. It follows from (3.36) and(3.35) that

$$\hat{L}^A(P \circ P) = (P \circ P)\hat{L}^A \tag{3.37}$$
$$\hat{W}(P \circ P) = (P \circ P)\hat{W}$$

The Hopf algebra $\hat{\mathcal{U}}$ and its module P^a can be viewed as a q-deformation of the Poincare algebra. Together, it does not have a Hopf algebra structure, however.

Of special interest is a $\hat{\mathcal{U}}$ model that is generated by two four-vector P^a and X^a whose algebra can be considered as a q-deformation of the Heisenberg algera. Its structure has been studied in detail in [10].

q-deformed Heisenberg algebra:

$$P^a X^b - \frac{1}{q^2}\left(R_{II}^{-1}\right)^{ab}{}_{cd} X^c P^d = -\frac{i}{2}\Lambda^{-\frac{1}{2}}\left\{\left(1+q^4\right)\eta^{ab}U + q^2\left(1-q^4\right)V^{ab}\right\} \tag{3.38}$$

The quantities $(R_{II}^{-1})^{ab}{}_{cd}$ and η^{ab} are defined in the Appendix. The element $\Lambda^{-\frac{1}{2}}$ is a "scaling" operator:

$$\Lambda^{-\frac{1}{2}} X^a = q X^a \Lambda^{-\frac{1}{2}} \tag{3.39}$$
$$\Lambda^{-\frac{1}{2}} P^a = \frac{1}{q} P^a \Lambda^{-\frac{1}{2}}$$
$$\overline{\Lambda^{\frac{1}{2}}} = q^4 \Lambda^{-\frac{1}{2}}$$

The element V^{ab} is a q-deformed angular momentum in the four-dimensional X^a plane. It has a Pauli decomposition:

$$V^{A0} = R^A + q^2 S^A \tag{3.40}$$
$$V^{0A} = -q^2 R^A - S^A$$
$$V^{AB} = \varepsilon_C{}^{AB}\left(R^C - S^C\right)$$
$$V^{00} = 0$$
$$\overline{R^A} = -g_{AB} S^B \qquad \overline{S^A} = -g_{AB} R^B$$

The element U is related to the Casimir operator:

$$U^2 - 1 = \frac{1}{2} q^4 [2]^2 \lambda^2 \left(R \circ R + + S \circ S\right) \tag{3.41}$$
$$\overline{U} = U$$

As is always the case, the representations of angular momentum are restricted. Thus R^A and S^A will not generate a full q-Lorentz algebra.
In [10] it was shown that this restriction leads to the equation

$$R \circ R = S \circ S \tag{3.42}$$

From the point of view of our $\hat{\mathcal{U}}$ algebra, the restriction is

$$\hat{U} = \hat{U}' \tag{3.43}$$

This, due to (1.4), leads to

$$\hat{R} \circ \hat{R} = \hat{S} \circ \hat{S} \tag{3.44}$$

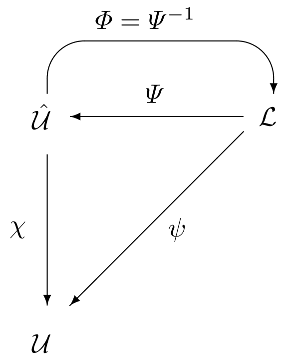
and due to (3.28) it leads to:

$$0 = g_{AB} P^A \left(R^B - q^2 S^B\right) \tag{3.45}$$
$$0 = P^0\left(S^A - q^2 R^A\right) - \varepsilon_{CB}{}^A P^B (R^C + S^C)$$

These are exactly the relations that were found in [10] and that generalize the fact that angular momentum is orthogonal to the momentum and to the coordinates. The statement for the coordinators is true here as well, P^a can be replaced by X^a in (3.45).

It should be noted that the factorization χ (3.43) is an algebra morphism from $\hat{\mathcal{U}}$ to an algebra that we shall call $\mathcal{U}$, but it is not a Hopf algebra morphism, $\Delta(\hat{U})$ and $\Delta(\hat{U}')$ do not coincide.

Our result can be represented by the following commuting diagramme of algebra morphisms:



A Metric, ε-tensor, R-martices

In the present paper we used the conventions of [10].

q-numbers are defined as:

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} \tag{A.46}$$

The nonvanishing entries of the q-deformed metric tensor are:

$$g_{AB} : g_{33} = 1, \tag{A.47}$$
$$g_{+-} = -q, \quad g_{-+} = -\frac{1}{q};$$
$$g^{AB} : g^{33} = 1, \tag{A.48}$$
$$g^{+-} = -q, \quad g^{-+} = -\frac{1}{q};$$

$$X \circ Y = g_{AB} X^A Y^B = X^3 Y^3 - q X^+ Y^- - \frac{1}{q} X^- Y^+$$
$$X_A = g_{AB} X^B, \quad X^A = g^{AB} X_B$$

The nonvanishing entries of the q-deformed ε- tensor are:

$$\varepsilon_{CB}{}^A : \varepsilon_{33}{}^3 = 1 - q^2, \tag{A.49}$$

$$\varepsilon_{+-}{}^{3} = q, \quad \varepsilon_{-+}{}^{3} = -q, \quad \varepsilon_{+3}{}^{+} = 1,$$
$$\varepsilon_{3+}{}^{+} = -q^2, \quad \varepsilon_{3-}{}^{-} = 1, \quad \varepsilon_{-3}{}^{-} = -q^2.$$

Raising and lowering of an index (shown only for the middle index, valid for the other indices as well) is done according to:

$$\begin{aligned} \varepsilon_{C}{}^{AB} &= g^{AD}\varepsilon_{CD}{}^{B} \\ \varepsilon_{CD}{}^{B} &= g_{DA}\varepsilon_{C}{}^{AB} \end{aligned} \tag{A.50}$$

Properties of the ε-tensor:

$$\begin{aligned} \varepsilon_{AB}{}^{C} &= g^{XC}\varepsilon_{XAB} \\ \varepsilon_{ABC} &= g_{XA}\varepsilon_{BC}{}^{X} \\ \varepsilon^{ABX}\varepsilon_{CDX} &= \varepsilon_{X}{}^{AB}\varepsilon_{CD}{}^{X} \\ &= \varepsilon^{XAB}\varepsilon_{XCD} \\ \varepsilon_{ABX}\varepsilon_{CD}{}^{X} &= q^2 g_{DA}g_{CB} - q^2 g_{BA}g_{DC} + \varepsilon_{AXD}\varepsilon_{BC}{}^{X} \\ \varepsilon_{C}{}^{AB}\varepsilon_{BA}{}^{D} &= (1+q^4)\delta_C^D \\ \varepsilon^{ABD}\varepsilon_{BAC} &= (1+q^4)\delta_C^D \\ g^{BA}\varepsilon_{ABX} = 0, &\quad g^{AB}\varepsilon_{ABX} = 0 \end{aligned} \tag{A.51}$$

$$\begin{aligned} \overline{\varepsilon_{AB}{}^{C}} &= \varepsilon^{BAX}g_{XC} \\ \overline{g_{AB}} &= g^{AB} \end{aligned} \tag{A.52}$$

The Euclidean $\hat{R}$-matrix:

$$\hat{R}^{AB}{}_{CD} = \delta_C^A\delta_D^B - \frac{1}{q^4}\varepsilon_{X}{}^{AB}\varepsilon_{DC}{}^{X} - \frac{q^2-1}{q^4}g^{AB}g_{CD} \tag{A.53}$$

q-deformed Minkowski metric:

$$\begin{aligned} \eta_{ab} &: \eta_{00} = -1, \quad \eta_{AB} = g_{AB}, \\ &\quad \eta_{0A} = 0, \quad \eta_{A0} = 0; \\ \eta^{ab} &: \eta^{ab} = \eta_{ab}; \end{aligned} \tag{A.54}$$

$$\begin{aligned} X \bullet Y = \eta_{ab}X^aY^b &= -X^0Y^0 + \\ &+ X^3Y^3 - qX^+Y^- - \frac{1}{q}X^-Y^+ \\ X_a = \eta_{ab}X^b, &\quad X^a = \eta^{ab}X_b \end{aligned}$$

More relations can be found in [10] [11].

B Useful relations in $\hat{\mathcal{U}}$

We give some relations which are needed for explicit calculations. Equivalent relations in $\mathcal{U}$ are obtained by setting $\hat{U}' = \hat{U}$. For more relations see [11].

$$\begin{aligned} \hat{Z}^A &= \varepsilon_{CB}{}^{A}\hat{R}^B\hat{S}^C \\ &= -\frac{1}{q^2}\varepsilon_{CB}{}^{A}\hat{S}^B\hat{R}^C \end{aligned} \tag{B.55}$$

$$\hat{R}\circ\hat{S} = \frac{1}{q^4}\hat{S}\circ\hat{R} \tag{B.56}$$

$$\begin{aligned} \hat{R}\circ\hat{Z} &= \frac{1}{q^2+1}\hat{U}\left(\hat{R}\circ\hat{S}\right) \\ \hat{S}\circ\hat{Z} &= \frac{q^2}{q^2+1}\hat{U}'\left(\hat{R}\circ\hat{S}\right) \\ \hat{Z}\circ\hat{R} &= -\frac{q^2}{q^2+1}\hat{U}\left(\hat{R}\circ\hat{S}\right) \\ \hat{Z}\circ\hat{S} &= -\frac{1}{q^2+1}\hat{U}'\left(\hat{R}\circ\hat{S}\right) \end{aligned}$$

$$\begin{aligned} \hat{R}^A\hat{S}^B &= q^2\hat{S}^A\hat{R}^B \\ &\quad +\varepsilon_{C}{}^{AB}\hat{Z}^C - \frac{q^2-1}{q^2}g^{AB}\left(\hat{S}\circ\hat{R}\right) \\ \hat{S}^A\hat{R}^B &= \frac{1}{q^2}\hat{R}^A\hat{S}^B - \frac{1}{q^2}\varepsilon_{C}{}^{AB}\hat{Z}^C + \left(q^2-1\right)g^{AB}(\hat{R}\circ\hat{S}) \end{aligned} \tag{B.57}$$

$$\begin{aligned} \varepsilon_{CB}{}^{A}\hat{R}^B\hat{Z}^C &= q^2\hat{S}^A\left(\hat{R}\circ\hat{R}\right) \\ &\quad -q^2\hat{R}^A\left(\hat{R}\circ\hat{S}\right) + \frac{1}{q^2+1}\hat{U}\hat{Z}^A \\ \varepsilon_{CB}{}^{A}\hat{S}^B\hat{Z}^C &= -\hat{R}^A\left(\hat{S}\circ\hat{S}\right) + \hat{S}^A(\hat{S}\circ\hat{R}) - \frac{1}{q^2+1}\hat{U}'\hat{Z}^A \\ \varepsilon_{CB}{}^{A}\hat{Z}^B\hat{R}^C &= -\left(\hat{R}\circ\hat{R}\right)\hat{S}^A + \left(\hat{S}\circ\hat{R}\right)\hat{R}^A + \frac{1}{q^2+1}\hat{U}\hat{Z}^A \\ \varepsilon_{CB}{}^{A}\hat{Z}^B\hat{S}^C &= q^2\left(\hat{S}\circ\hat{S}\right)\hat{R}^A - q^2\left(\hat{R}\circ\hat{S}\right)\hat{S}^A \\ &\quad -\frac{1}{q^2+1}\hat{U}'\hat{Z}^A \end{aligned} \tag{B.58}$$

$$\begin{aligned} \hat{R}^A\left(\hat{R}\circ\hat{S}\right) &= \frac{q^2-1}{q^4}\left(\hat{R}\circ\hat{R}\right)\hat{S}^A \\ &\quad +\frac{1}{q^2}\left(\hat{R}\circ\hat{S}\right)\hat{R}^A + \frac{1}{q^4(q^2+1)}\hat{U}\hat{Z}^A \\ \hat{S}^A\left(\hat{R}\circ\hat{S}\right) &= -\frac{q^2-1}{q^2}\left(\hat{S}\circ\hat{S}\right)\hat{R}^A + q^2\left(\hat{R}\circ\hat{S}\right)\hat{S}^A \\ &\quad +\frac{1}{q^2\left(q^2+1\right)}\hat{U}'\hat{Z}^A \end{aligned} \tag{B.59}$$

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